Eikonal Equation

This is another form of Fermat's principle which states that if $S(R)$ represents the the phase front surface, then

$$|\nabla S|^2 = n^2$$ where ∇ is the gradient operator

Now in graded index fibre, we consider that a quasi plane wave travels, which is shown as

$$E(R) = e(R) \exp[-jk\, s(R)] \qquad R = (x, y, z) \quad R = (r, \phi, z)$$

Here $e(R)$ and $S(R)$ vary slowly in comparison with wavelength λ. from above we know that S is also related to refractive index from our previous developments we know that

dependence on ϕ is as $\exp(-j\nu\phi)$

and dependence on z is as $\exp(-j\beta z)$

on r as $k_r(r)$

finally we formulate the dependence on r

$$\exp\left(-j\int_0^r k_r(r)\,dr\right), \text{ hence}$$

$$-jkS(R) = -j\int_0^r k_r(r)\,dr - j\nu\phi - j\beta z$$

Now apply $\nabla = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z}$ and

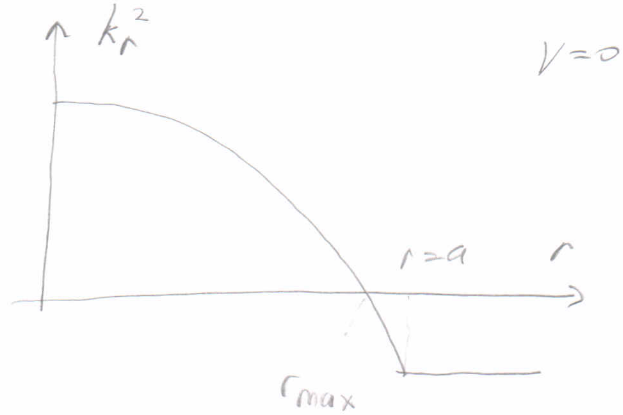
set $k^2|\nabla S(R)|^2 = n^2(r)k^2$

$$k_r^2(r) = n^2(r)k^2 - \beta^2 - \frac{\nu^2}{r^2}$$

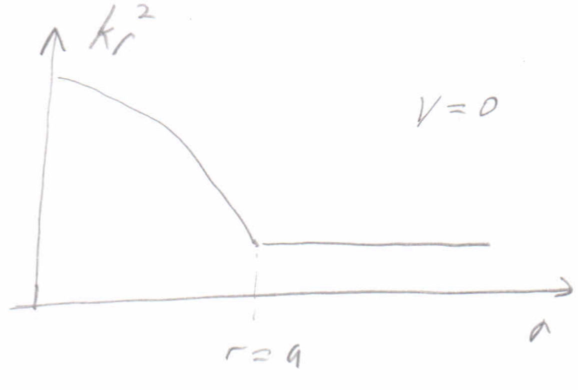
The wave travels within the core where

k_r^2 is positive

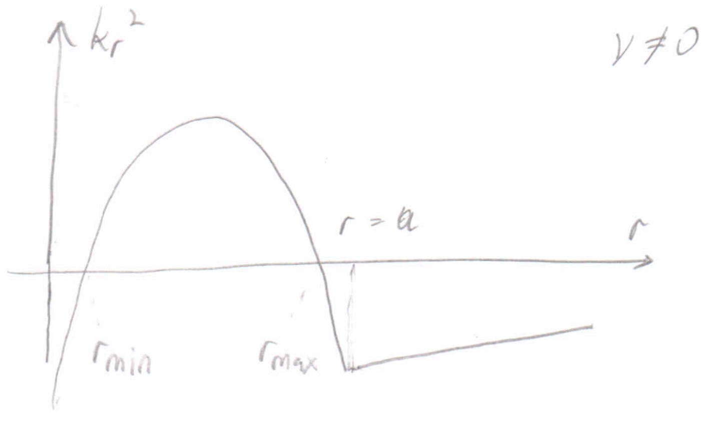
outside these regions k_r^2 is negative or k_r is imaginary, so the wave decay exponentially there. In general we have the following picture



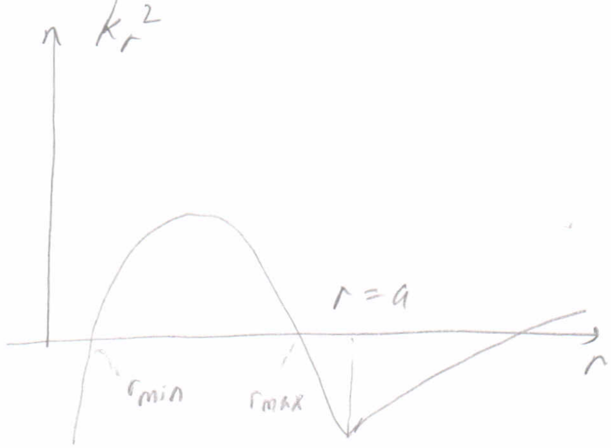
$\Rightarrow$ Meridional
Bound (Reflecting)
Guided



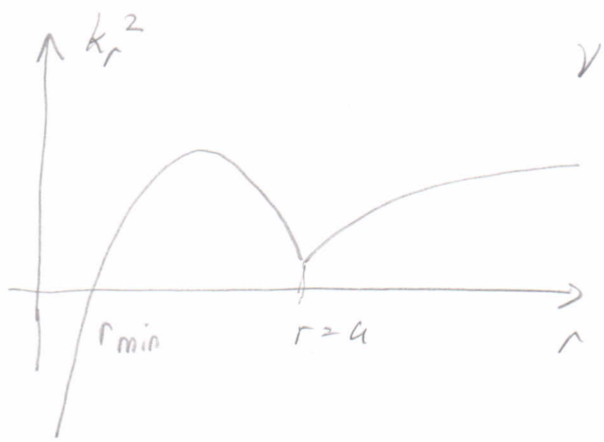
$\Rightarrow$ Meridional
Refracting



Skew Bound
Guided



$\nu \neq 0$ Skew Refracting (Leaky)



$\nu \neq 0$ Skew refracting

On the other hand, by taking the integral of k_r between r_{min} and r_{max} (which are found by setting $k_r = 0$ and retrieving roots)

$$2 \int_{r_{min}}^{r_{max}} k_r \, dr = 2\pi m \quad , \; m \rightarrow \text{integer}$$

which means that radial path length must correspond to a phase shift of $m \times 2\pi$

Solving this integral will give us for

$n(r) = n_1 \left[1 - 2\Delta (r/a)^2 \right]^{1/2}$, i.e. quadratic profile

$$\frac{a(k^2 n_1^2 - \beta^2)}{4 k n_1 \sqrt{2\Delta}} - \frac{V}{2} = m$$

From here it is possible to find β's in a simple manner, given m and such that

$$\beta = [kn_1^2 - 4kn_1\sqrt{2\Delta}(m+\nu/2)/a]^{1/2}$$

From this equation we can determine β (propagation constant) of graded index fibres and also check which β will correspond to which class of ray listed on pp. 12, 13